THE UNEXPECTED COSTS OF REBALANCING AND HOW TO ADDRESS THEM

Nick Granger, Douglas Greenig, Campbell Harvey, Sandy Rattray, David Zou

June 2014

OVERVIEW

• Investors commonly rebalance their portfolios to maintain desired asset allocations as prices change. Rebalancing often aims for a constant-mix portfolio, such as allocating 60% of the portfolio to equities and 40% to bonds.

• Since diverging asset returns change the portfolio allocation, some form of rebalancing is needed to avoid extreme allocations. A 60/40 portfolio, for example, can easily drift over a couple of years to a 75/25 during an equity bull market. Helpfully, rebalancing can earn a ‘rebalancing premium’ in some range-bound market environments.

• Rebalancing, however, may impose unexpected costs on investors by magnifying drawdowns when there are pronounced divergences in asset performance. Most portfolios are dominated by equity risk, and equity returns historically have exhibited fat downside tails. Regularly rebalanced portfolios not only inherit these properties but actually exacerbate the drawdowns.

• A momentum overlay can reduce the risks associated with rebalancing by improving the timing of the rebalance. The rebalanced portfolio in combination with the momentum overlay potentially reduces drawdowns and improves risk properties during crisis periods where stock and bond market returns diverge, while maintaining relatively stable portfolio weights over the long term.
## CONTENT

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>3</td>
</tr>
<tr>
<td>Fixed weights vs. drift weights</td>
<td>3</td>
</tr>
<tr>
<td>Why does fixed-weight rebalancing have higher drawdowns?</td>
<td>4</td>
</tr>
<tr>
<td>Momentum has the opposite characteristics to fixed-weight rebalancing</td>
<td>5</td>
</tr>
<tr>
<td>Momentum to the rescue?</td>
<td>5</td>
</tr>
<tr>
<td>Why don’t rebalancing and momentum just cancel?</td>
<td>6</td>
</tr>
<tr>
<td>Conclusion</td>
<td>7</td>
</tr>
<tr>
<td>Appendix</td>
<td>8</td>
</tr>
<tr>
<td>1. A1 Fixed-weight rebalance of one asset and cash</td>
<td></td>
</tr>
<tr>
<td>2. A2 Trend following</td>
<td></td>
</tr>
<tr>
<td>3. A3 Combining Momentum and Rebalancing</td>
<td></td>
</tr>
<tr>
<td>4. A4 Fixed-weight rebalancing of two assets</td>
<td></td>
</tr>
</tbody>
</table>

The authors like to thank Anthony Ledford and Darrel Yawitch for their comments and contributions.

www.ahl.com
www.man.com
www.oxford-man.ox.ac.uk
INTRODUCTION

In this paper, we explore some surprising properties of rebalancing and show how a momentum overlay can improve risk and drawdown properties without reducing expected returns in a constant-mix portfolio.

Constant-mix strategies, such as allocating 60% of the value of a portfolio to equities and 40% to bonds, are used extensively by pension funds and other long-term investors. In such strategies, the investor periodically rebalances the portfolio so that each asset class is a constant fraction of portfolio value rather than allowing the allocation to drift as asset prices change.

Constant-mix strategies have intuitive motivations, but we believe key properties of these strategies are poorly understood by many investors. In particular, rebalancing can magnify drawdowns when there are pronounced divergences in asset performance. Such divergences are usually driven by equities, and in late 2008 and early 2009, some rebalanced strategies underperformed passive strategies by hundreds of basis points.

In traditional 60/40 portfolios, the vast majority of the risk (ca. 85%) comes from the allocation to equities. Equity indices, in addition to generally having much higher volatilities than bond indices, have fat downside tails, i.e. large negative returns, and sharp drawdowns occur more frequently than large positive returns. Constant-mix strategies not only inherit these properties, but actually exacerbate the drawdowns.

We demonstrate how the opposing return profile of a momentum overlay can help to mitigate the added risk introduced by the rebalancing process. Even without any assumptions of positive performance, momentum acts to improve the risk and drawdown properties of the constant-mix portfolio according to both theoretical and historical analyses.

We provide a technical appendix which gives analytic explanations for the results described in the main body of the paper. This appendix provides the theoretical arguments to elucidate how the rebalancing process can change the risk properties of the portfolio and how a momentum overlay can help. The appendix is provided principally to demonstrate that the results in this paper are of an analytic nature and are not dependent on the particular realisation of history seen in the empirical data. If one can take this on trust, reading the appendix is purely optional.

FIXED WEIGHTS VS. DRIFT WEIGHTS

A passive buy-and-hold strategy is an alternative to a constant-mix strategy. Under buy-and-hold, one simply buys a portfolio (e.g. 60% equities/40% bonds) and holds it without trading for a period of time. The capital allocation within the portfolio will obviously drift as market prices change, but the investor remains passive. In contrast, the active rebalancing in the constant-mix strategy will restore the target fixed weights. We shall also call the buy-and-hold approach a drift-weight portfolio, to be contrasted with fixed weights under rebalancing.

The motivations behind a constant-mix strategy are straightforward. First, drift weights may become extreme as assets diverge. In 2013, the S&P500 delivered a total return of 31.9%, while the S&P 7-10 Year US Treasury bond index declined by 6.1%. A 60/40 portfolio would have drifted over 2013 to a 68/32 portfolio, and, with a repetition of these returns in 2014, a 74/26 portfolio. In practice, few if any investors remain passive indefinitely as drift weights become extreme. At some point, the investor is likely to adjust the weights. The constant-mix strategy sensibly puts this adjustment process on a regular schedule.

The so-called ‘rebalancing premium’ is a second motivation for the constant-mix strategy. The premium refers to the extra returns that the rebalancing process generates under certain circumstances. Let’s consider how such profits might arise. Divergent asset performance will cause the weights to differ from the target allocation. To restore the desired allocation, the investor must buy some of the underperforming assets and sell some of the outperformers. ‘Buying low and selling high’ has an intuitive appeal, and provided there is mean reversion in relative asset performance, rebalancing to fixed weights may generate positive returns.

On the other hand, when assets diverge strongly over time, fixed-weight rebalancing may cause losses relative to buy-and-hold as the rebalancing continues to increase the allocation to the underperforming asset. Figures 1 and 2 show the performance of constant-mix strategies vs. drift-weights during two bear markets in equities: 2008/9 and 2000/2. The portfolios start with 60% allocations to the S&P500 and 40% allocations to the 10-year US Treasury bond. Rebalancing is done monthly. In both periods, rebalancing exacerbated losses and increased the drawdowns by about 500 bps. Although these were clearly exceptional periods, and do not reflect typical performance, they demonstrate the extra risk inherent in the rebalancing process. In the event, markets recovered in subsequent years, and the constant mix portfolio eventually caught up with the passive portfolio.

Figure 1: Rebalancing magnified drawdown in 2008-2010 Cumulative returns to 60/40 portfolios of the S&P500 and 10-year US Treasury

1. Over the last ten years the average annualised volatility of the S&P500 from daily closes has been 20.6%. The average volatility of the 10 year US Treasury bond has been 6.3%. Source: Bloomberg.
2. Source: Bloomberg. The dates chosen were based on the research and assessment of the portfolio as explicit start and end dates for these events were not available. To a certain extent, the start and end dates of such events are subjective and different sources may suggest different date ranges, leading to different performance figures. As a consequence, they give no indication of likely performance.

The performance difference between fixed weights and drift weights shows the characteristic return profile (sometimes called ‘negative convexity’) of an investor who has sold both put and call options (on the relative performance of stocks and bonds). This profile shows positive, but limited, returns under small moves and large negative returns under large moves. The driver is the divergence in stock and bond returns because differences in return change the allocation. Viewing this another way, when assets perform equally, there is no need for rebalancing trades, even if the returns are large.

To illustrate, Figure 3 shows the difference in one-year returns between fixed-weight rebalancing and drift weights using both empirical (panel A) and simulated (panel B) data. The empirical analysis uses overlapping one-year returns for 60/40 strategies with monthly rebalancing on the S&P500 and 10-year US Treasuries, using data from 1990 to present. The simulation takes two asset returns using lognormal distributions with volatilities of 20% and 6% and a constant -20% correlation. The constant-mix strategy often earns a small rebalancing premium, but at the risk of occasionally sharply underperforming the buy-and-hold strategy.

**Figure 3: Fixed weights vs. drift weights: Performance difference over one year**

**A. One-year returns on empirical data, fixed-weight rebalancing minus drift weights**

**B. One-year returns on simulated data, fixed-weight rebalancing minus drift weights**


---

**WHY DOES FIXED-WEIGHT REBALANCING HAVE HIGHER DRAWDOWNS?**

Buying losers and selling winners is the essence of rebalancing to fixed weights. Market dynamics ultimately determine whether such a trading strategy is profitable. A trend-less environment, especially one characterised by mean reversion, would clearly be friendly to this strategy – losers would bounce back, high-flyers would come down to earth, and rebalancing investors would gain from their trades.

But trending markets, e.g. where equities keep losing relative to bonds, pose a problem for rebalancing. A sustained trend means one buys an asset on the way down, ‘catching a falling knife’ in the popular parlance. Trending (or ‘time-series momentum’) is in fact a well-established property of markets over the last century and remains an important trading signal. One ignores trends at one’s own peril.

Figures 1-3 show rebalancing is no free lunch but represents an implicit bet against diverging asset performance. The cost occurs when markets trend and rebalanced portfolios experience deeper drawdowns than buy-and-hold portfolios. For small values of stock-bond return divergence, fixed-weight rebalancing modestly outperforms buy-and-hold, as one can see in Figure 3. But for large divergences, the underperformance of the constant mix is marked.

In the Appendix, we use analytic methods to derive the aforementioned risk properties of rebalancing and buy-and-hold portfolios. The derivation is a theoretical proof, meaning that these risk properties arise from the dynamic style of trading rather than any special properties of the underlying markets.
MOMENTUM HAS THE OPPOSITE CHARACTERISTICS TO FIXED-WEIGHT REBALANCING

Momentum trading, in contrast to rebalancing, involves the purchase of winners and sale of losers, while maintaining a given volatility exposure. We have previously shown in an article in Risk Magazine5 that momentum trading produces a positively convex and positively skewed distribution of returns. This result is also derived analytically. Remarkably, the positive convexity and skewness do not depend on strong assumptions about the underlying markets; but can be proved mathematically. The dynamic style of trading introduces the positive skewness, which translates into reduced drawdowns relative to passive strategies.

Figure 4 shows the pay-off profile of momentum trading on a single simulated asset. One-year asset returns are simulated using daily lognormal distributions for returns (with 20% annualised volatility and zero serial correlation). Then we use a moving average cross-over system, to be described in the next section. Intriguingly, it looks like the exact opposite of the profile for the performance difference between fixed- and drift-weights. One important distinction is that the x-axis of Figure 4 represents the absolute performance of the single asset whereas Figure 3 uses the relative performance of stocks and bonds. No fees or trading costs were imposed in the simulation. As shown in the above cited article, the shape of the payoff profile is a general property of momentum systems.

In Figure 5, we apply momentum to stocks and bonds individually, and graph the performance as a function of the divergence between stocks and bonds. Panel A uses overlapping one-year returns to the S&P500 and the 10-year US Treasury bond, while panel B uses simulated data as in Figure 3.

MOMENTUM TO THE RESCUE?

In our analysis, adding a momentum overlay to a constant-mix portfolio improves the risk properties of the portfolio. The concept of an overlay is familiar to many pension plans, where overlays are commonly used, particularly on the liability side. But on the asset side, momentum overlays can be used to hedge the drawdown risks mechanically induced by rebalancing. The overlay tends to gain from the very trends that hurt the constant-mix strategy, but imposes little cost in other environments.

This hedging property is seen in both empirical data and in simulations. Figure 6 illustrates the cumulative return of assets with fixed-weight rebalancing combined with a 10% momentum overlay, as a function of asset divergence.

1. ‘Momentum strategies offer a positive point of skew’ Risk, August 2012. 2. We chose 10% as a reasonable initial case to consider; we discuss other levels later.
The momentum overlay uses a moving average cross-over model employed at AHL. It has a target volatility of 10% and turns over positions, on average, four times a year. The model has a number of features – including applying a response function to the raw signal and scaling by volatility. While these features may improve performance, the essential story remains the same even for simpler momentum models.

In addition, Table 1 shows the impact of varying amounts of momentum allocation on a constant mix portfolio using empirical data. We see, for example, that increasing the momentum overlay modestly increases the Sharpe ratio and improves the skewness, at both rebalancing frequencies. Moreover the maximum drawdown is reduced materially, depending on the amount of the overlay.

The effect of adding the momentum overlay is to neutralise the negative tails from rebalancing seen in Figure 3.

**WHY DON’T REBALANCING AND MOMENTUM JUST CANCEL?**

Doesn’t adding momentum to a constant-mix portfolio merely get you back to where you started, a passive buy-and-hold portfolio?

In our view the answer is no. Consider the S&P500 and 10-year Treasuries. Figure 7 shows the portfolio weight on equities at each point in time, since 2000, under different portfolio construction strategies. The blue line comes from a 60/40 monthly rebalanced portfolio, the grey line shows the same portfolio with a 20% momentum overlay, and the brown line illustrates a simple drift-weight portfolio starting at the same 60/40 allocation. Over short time-frames, the portfolio with the momentum overlay can differ quite markedly from the standard rebalanced portfolio. Over the long-run, however, the portfolio with the overlay tracks the long-term desired allocation. In contrast, the drift-weight portfolio drops to equity allocations as low as 25% by 2009. The rebalanced portfolio with momentum overlay, on the other hand, stays between 40% and 75%, while the rebalanced portfolio without overlay is locked around 60% equities.

**Table 1: Impact of momentum overlays on constant-mix portfolios**

<table>
<thead>
<tr>
<th>Rebalance frequency</th>
<th>Momentum allocation</th>
<th>Annualised return</th>
<th>Annualised volatility</th>
<th>Sharpe Ratio</th>
<th>Skewness</th>
<th>Correlation to rebalanced portfolio without overlay</th>
<th>Max drawdown</th>
<th>Max drawdown depth</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monthly</td>
<td>0%</td>
<td>3.5%</td>
<td>9%</td>
<td>0.40</td>
<td>-0.99</td>
<td>1.00</td>
<td>-34.4%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10%</td>
<td>4.2%</td>
<td>9%</td>
<td>0.45</td>
<td>-0.80</td>
<td>1.00</td>
<td>-31.9%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>20%</td>
<td>4.8%</td>
<td>10%</td>
<td>0.50</td>
<td>-0.62</td>
<td>0.98</td>
<td>-29.6%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>40%</td>
<td>5.8%</td>
<td>10%</td>
<td>0.58</td>
<td>-0.30</td>
<td>0.93</td>
<td>-25.3%</td>
<td></td>
</tr>
<tr>
<td>Quarterly</td>
<td>0%</td>
<td>3.7%</td>
<td>9%</td>
<td>0.42</td>
<td>-0.90</td>
<td>1.00</td>
<td>-33.0%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10%</td>
<td>4.3%</td>
<td>9%</td>
<td>0.47</td>
<td>-0.72</td>
<td>1.00</td>
<td>-29.5%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>20%</td>
<td>4.8%</td>
<td>9%</td>
<td>0.52</td>
<td>-0.50</td>
<td>0.98</td>
<td>-27.4%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>40%</td>
<td>5.9%</td>
<td>10%</td>
<td>0.60</td>
<td>-0.26</td>
<td>0.93</td>
<td>-23.6%</td>
<td></td>
</tr>
</tbody>
</table>

Source: Reuters, Man calculations. Date range: January 2000 to February 2014. Exclusive of fees and transaction costs.
In any case, a passive buy-and-hold portfolio may not be a realistic option for most investors over the long-term, since drift weights may become extreme. Rebalancing, whether on a regular schedule or when drift weights become intolerably imbalanced, is nearly inevitable. The pertinent question, then, is what investors should do about the increased drawdown risk from this rebalancing. Our analysis indicates that a momentum overlay allows one to maintain a sensible rebalancing schedule without increased drawdown risk.

While rebalancing and momentum signals treat outperforming (and underperforming) assets in broadly opposite ways, with rebalancing selling winners and momentum buying them, there are important differences in the trading. The momentum overlay, operating on a daily time-scale, effectively acts as a timing strategy around the monthly or quarterly rebalance. For example, the overlay will resist buying too aggressively into equities in times of market stress (easing the psychological pressure on the rebalancer). But, as divergences in asset performance stabilise, the momentum overlay will cut out and allow the full rebalance to occur. That is instead of undoing the rebalance, the momentum overlay, acting on a shorter-time scale, seeks to improve the timing of the rebalance. Asset returns have historically displayed positive autocorrelation, especially in times of market stress, and the momentum overlay allows the portfolio to take this into account as part of the rebalancing process.

The positive impact of the overlay on portfolio performance can be seen in Figures 8, 9 and 10 and Table 1. Figure 8 shows the cumulative returns of a 60/40 constant mix portfolio. Table 1 varies the amount of overlay. Performance and risk properties improve as the overlay is increased.

In Figures 9 and 10, we see how the overlay helps to mitigate the negative impact of rebalancing. Indeed, we see that the overlay reduces the drawdowns in both 2001-2004 and 2008-2009 and nearly replicates drift-weight performance. In the case of the latter episode, fixed weight rebalancing plus the overlay outperforms the other methods by the summer of 2009.

**CONCLUSION**

Rebalancing is sometimes touted as a costless way to enhance returns by means of rebalancing premium. While a premium can be earned when relative asset prices do not diverge strongly, diverging asset prices can lead to marked underperformance. Portfolio rebalancing can exacerbate drawdowns during major trends, as in 2000-2002 and 2008-2009. The performance loss from rebalancing reached, in those examples, around 500 bps at the bottom (Figure 1 and 2).

The driver behind the worst drawdowns is the fact that rebalancing represents a kind of anti-momentum trading in which one buys underperforming assets and sells outperforming assets. The active trading associated with rebalancing introduces ‘negative convexity’ – a return profile in which large divergences in asset performance can cost the investor a lot of money in return for a small rebalancing premium when asset prices do not diverge.

---

7. The dates chosen were based on the research and assessment of the portfolio as explicit start and end dates for these events were not available. To a certain extent, the start and end dates of such events are subjective and different sources may suggest different date ranges, leading to different performance figures. As a consequence, they give no indication of likely performance.
Rebalancing, however, is here to stay. Passive portfolio construction in which weights are allowed to drift is not a realistic alternative, because allocations can get extreme after only a few years. So one must rebalance in some way and must therefore deal with the effects of rebalancing on drawdowns and tail risk.

Adding a momentum overlay can help to mitigate the added risk introduced by the rebalancing process.

• First, a momentum overlay can introduce an opposing return profile, generating gains when trends are strong and asset performance diverges.

• Second, momentum operates on a much shorter time-scale than the rebalancing process and effectively acts as a timing strategy around the monthly or quarterly rebalance.

• Third, both theoretical and empirical analyses indicate that introducing a momentum overlay to a rebalanced portfolio reduces drawdowns and improves risk-adjusted performance.

• Finally, momentum trading itself has historically produced positive returns over long periods of time. 8

However, even without any assumptions of positive performance, our analysis shows that momentum acts to improve the risk and drawdown properties of the constant-mix portfolio.

APPENDIX

First, we derive analytic results for the fixed-weight rebalancing of a risky asset (e.g. a stock index) and cash. We show that the final payoff of the strategy is negatively convex in comparison with the drift weights case and has inferior drawdown characteristics.

Second, we derive analytic results for momentum trading and show that momentum returns are positively convex in the underlying.

Third, we look at combining a fixed-weight rebalanced portfolio with a momentum overlay and show how this overlay can reduce the negative convexity, worst returns and maximum drawdowns.

Finally, we introduce a second risky asset, say bonds. We derive similar but more complicated analytic results in this two-asset case.

A1 Fixed-weight rebalance of one asset and cash

We begin with the simpler case of rebalancing between a single risky asset and cash. In practice this simplification is representative of the more complex two-asset case because equity risk dominates that of bonds so cash is a reasonable proxy for bonds in this context. Later we examine the more complex case of two risky assets. Although the algebra is more involved the key ideas remain very similar.

We model the price of a single risky asset $S_t$ using a drift-less geometric Brownian motion:

$$\frac{dS_t}{S_t} = \sigma dW_t$$

$$S_t = S_0 e^{\mu t + \frac{\sigma^2}{2} t}$$

where $\sigma$ is the asset volatility and $W_t$ is a Wiener process.

Suppose our wealth at $t$ is $X_t$, with a fixed $m \in [0,1]$ invested in equity and $1-m$ in cash. Following a similar derivation to Perold and Sharpe,10 we obtain:

$$\frac{dX_t}{X_t} = m \frac{dS_t}{S_t}$$

Applying Ito’s lemma to log $X_t$ and taking the exponent, we obtain:

$$X_t = X_0 e^{m \mu t + \frac{1}{2} \sigma^2 t}$$

The final pay-off is proportional to $\left(\frac{X_t}{X_0}\right)^m$, i.e. the terminal stock price divided by the initial price, raised to the $m$th power.

Equation 2 shows the Taylor expansion of $X_t$ around $S_t = S_0$.

When $m < 1$, the term $\frac{(S_t - S_0)}{S_0}^2$ has a negative coefficient, which leads to negative convexity of the payoff as shown in Figure 11:

$$X_t = c X_0 \left(\frac{S_t}{S_0}\right)^m = c X_0 \left[1 + m \frac{(S_t - S_0)}{S_0} + \frac{1}{2} m (m-1) \frac{(S_t - S_0)^2}{S_0^2} + R_3\right]$$

where $c = e^{\frac{1}{2} \sigma^2 t}$

and $R_3 = o \left(\frac{S_t}{S_0} - 1\right)^3$ is the remainder term of the expansion.

8. ‘Momentum in Futures Market’, Norges Bank Investment Management, Discussion Note #1-2014. 9. The stochastic calculus used below necessarily assumes continuous rebalancing. In practice, and as described in the main the section of this paper, rebalancing takes place at discrete frequencies – e.g. monthly. Although the discrete rebalancing can be seen as an approximation of the continuous case derived here, our empirical and Monte Carlo results demonstrate that this approximation is valid (Figure 11). For ease of exposition we also assume the risk-free rate is zero throughout. 10. ‘Dynamic Strategies for Asset Allocation’ Financial Analysts Journal, 1988.
The partial derivative with respect to \( m \) of this expression for the skewness of the fixed weight rebalance is given by:

\[
\frac{\partial}{\partial m} \left[ \sqrt{e^{m^2 \sigma^2 t}} \right] = \frac{3 m \sigma^2 t e^{m^2 \sigma^2 t}}{\sqrt{e^{m^2 \sigma^2 t} - 1}}
\]

This term is always positive. Thus it is clear that skewness of the fixed weight portfolio decreases as \( m \) decreases. Therefore the rebalancing process (reducing \( m \) from \( 1 \)) reduces the skewness (which in this case results in a fatter left tail) in comparison to the original lognormal distribution of the drift-weight portfolio.

This is not surprising given the negatively convex payoff we showed in Eq. 2 and Figure 11.

These changes in skewness and convexity have implications in terms of drawdowns and worst returns compared to a simple drift weight (buy and hold) process. We demonstrate this via Monte-Carlo simulations below.

In Figure 12 we illustrate the distribution of final pay-offs under different levels of \( m \). The final pay-off of the dynamically rebalanced strategy has a fatter left tail in comparison to the drift-weight strategy. This is as we would expect due to the less favourable skewness properties of the dynamic strategy.

In Figure 13 we show the maximum drawdown experienced during the lifetime of the strategy. The drawdowns are larger for the dynamically rebalanced portfolio, as expected. These results are due to the negative convex exposure and reduced skewness from the dynamic rebalancing.

**Figure 12: Distribution of final pay-offs for \( m = 0.4, 0.6 \), 5-year horizon**
A2 Trend following

We have shown elsewhere\textsuperscript{11} that trend-following has positive skewness by construction. Intuitively this happens because trend following has natural stops: it reduces positions when it loses money and only builds positions when it makes money. Furthermore a trend following strategy has a pay-off convex in the underlying due to the embedded optionality.

The expected P&L of trading momentum on an asset $S$ from time $\theta$ to $t$ is a function of the terminal price $S_t$:

$$E(P&L_{\text{momentum}}|S_t) = Y_0\psi(t) \left( \ln \left( \frac{S_t}{S_0} \right) - \sigma^2 t \right)$$

where $Y_0$ is the initial risk allocation to the momentum strategy and $\psi(t)$ is a (positively valued) function on $t$, dependent the design of the signal.\textsuperscript{12} In this paper the momentum system is a simplified version of the moving average cross-over system in use at AHL. This formula makes explicit both the convexity of the momentum system and the optionality – including the time decay represented by the $\sigma^2 t$ term.

Figure 14 depicts a Monte-Carlo simulation of trading a moving-average cross-over momentum system on an asset $S_t$ following a geometric Brownian motion. The convexity of the payoff suggested by Eq. 3 is clear to see.

However Figure 14 shows the momentum return as a function of the log asset return. We can see the relationship with simple asset returns by simplifying Eq. 3 with a Taylor expansion:

$$E(P&L_{\text{momentum}}|S_t) = Y_0\psi(t) \left( \frac{S_t}{S_0} - 1 \right)^2 - \sigma^2 t + R_S$$

where $R_S = 0 \left( \frac{S_t}{S_0} - 1 \right)^3$ is the remainder term.

A3 Combining Momentum and Rebalancing

We now examine the performance of combining the rebalancing approach of section A1 with the trend-following strategy of section A2. We express the allocation to momentum as $Y_0 = kX_0$, i.e. the risk allocation to momentum is set as a fixed proportion $k$ of the initial wealth $X_0$.\textsuperscript{13} Combining Eq. 2 and Eq. 4 then gives:

$$E(P&L_{\text{fixed weight rebalance + momentum}}|S_t) = \left( cX_0 \left( m \left( \frac{S_t}{S_0} - 1 \right)^2 + \frac{1}{2}m(m-1)\left( \frac{S_t}{S_0} - 1 \right)^2 + R_S \right) - X_0 \right)$$

$$+ Y_0\psi(t) \left( \frac{S_t}{S_0} - 1 \right)^2 - \sigma^2 t + R_S$$

$$= X_0 \left( cm \left( \frac{S_t}{S_0} - 1 \right)^2 + \frac{1}{2}m(m-1)\left( \frac{S_t}{S_0} - 1 \right)^2 + \xi(t) + R_S \right)$$

where $R_S = 0 \left( \frac{S_t}{S_0} - 1 \right)^3$ collects all remainder terms and

$$\xi(t) = c - 1 - k\psi(t)\sigma^2 t$$

Since the above expression has overall mean approximately zero, and recalling that $c = e^{\frac{1}{2}(m-m^2)\sigma^2 t}$, we see that $\xi(t)$ is a deterministic time-decay offset to the quadratic $\left( \frac{S_t}{S_0} - 1 \right)^2$ term. This equation therefore pinpoints the theoretical basis for momentum offsetting the negative convexity of rebalancing that we showed in Figure 6.

\textsuperscript{11} ‘Momentum strategies offer a positive point of skew’ Risk, August 2012. \textsuperscript{12} The derivation of this is out of the scope of this paper and will be published in future paper. \textsuperscript{13} In reality rather than having the momentum allocation as a fixed proportion of the initial wealth, the momentum allocation itself would be dynamically rebalanced so as always to account for a fixed proportion $k$ of total current wealth. This is very much a second order effect and we work with a constant momentum allocation to keep the mathematics tractable.
In Figure 15 we show together the pay-offs of rebalancing and momentum. Note the change in the x-axis from Figure 14 where we have moved from log asset returns to simple percentage returns (as given by Eq. 4). We see that the rebalanced stocks-and-cash strategy has an upward slope with respect to the terminal stock value and a negatively convex shape. In contrast, momentum on $S_t$ has positive convexity.

Figure 15: The final pay-off of momentum and dynamic rebalancing with $m=0.6$

By offsetting the negative convexity of the rebalance process, the addition of a momentum system to a fixed-weight rebalanced portfolio (in the one asset case) produces distributions with a thinner left tail.

These facts translate directly into smaller drawdowns. Figures 16 and 17 illustrate the results of Monte-Carlo simulations over one and five year horizons. In both figures the addition of momentum reduces the left tail, thickens the right tail and increases the positive skewness of the P&L. The potential maximum drawdown is also noticeably reduced on the left tail with increasing allocation to momentum. In other words the fixed weight rebalance portfolio is less likely to experience extremes of drawdowns with allocation to momentum.14

Figure 16: Terminal pay-off and worst drawdowns for fixed-weight rebalancing and momentum, 1-year horizon
A. Profit and loss

Figure 17: Terminal pay-off and worst drawdowns for fixed-weight rebalancing and momentum, 5-year horizon
A. Profit and loss

14. Contrary to first impressions the mean of the return distribution is unchanged, with the shift of the mode to the left compensated for with the reduced left tail and increased right tail. Indeed since all of these Monte-Carlo simulations are Martingales the mean must by definition be zero.
A4 Fixed-weight rebalancing of two assets

In this section we derive analytic results for the two correlated asset case. The main difference with the one asset case is that the fixed-weight rebalance induces negative convexity in the divergence between the assets as well as in the individual asset returns.

We consider the case of two risky assets, $S_t$ and $B_t$, both of which are driftless with constant volatility. This simplifies the model and highlights the key points. By taking driftless assets we avoid introducing a constant trend which would act to bias results in favour of momentum.

Although both $S_t$ and $B_t$ are assumed to be driftless, one can nevertheless extend these results to a portfolio of stocks and bonds. With appropriate numeraire (e.g. money market account) the price processes of stocks and bonds are driftless in the presence of non-zero interest rates.

In addition, within a stock-bond portfolio, the duration of the bond allocation is generally maintained around a certain target. Hence the constant volatility assumption for $B_t$ is reasonable in this case.

In any case – since the purpose of this paper is compare various forms of fixed-weight and drift-weight portfolio – the differences between these portfolios introduced by non-zero drift and any stochastic rate element are not our primary focus.

Under the above assumptions we have:

\[
\frac{dS_t}{S_t} = \sigma_d dW_{S,t} \quad \text{and} \quad \frac{dB_t}{B_t} = \sigma_B dW_{B,t}
\]

Denoting the return correlation between $S_t$ and $B_t$ by $\rho$, we have that the total wealth $X_t$ satisfies

\[
\begin{align*}
\frac{dX_t}{X_t} &= \frac{dS_t}{S_t} + n \frac{dB_t}{B_t} \\
&= m \frac{dS_t}{S_t} + n \frac{dB_t}{B_t}
\end{align*}
\]

where $m$ and $n$ represent the investment weights in $S$ and $B$.

Similarly to before, applying Ito’s lemma to $\ln X_t$, we obtain

\[
\begin{align*}
\ln X_t &= \frac{1}{X_t} \frac{dX_t}{X_t} - \frac{1}{2} \frac{dX_t}{X_t}^2 \\
&= \frac{m \sigma_d dW_{S,t} + n \sigma_B dW_{B,t}}{X_t} - \frac{1}{2} \left( m^2 \sigma_d^2 + n^2 \sigma_B^2 + 2 \rho mn \sigma_d \sigma_B \right) dt
\end{align*}
\]

and hence

\[
X_t = X_0 \exp \left[ m \sigma_d W_{S,t} + n \sigma_B W_{B,t} + \frac{1}{2} \left( m^2 \sigma_d^2 + n^2 \sigma_B^2 + 2 \rho mn \sigma_d \sigma_B \right) t \right] = X_0 \left( \frac{S_t}{S_0} \right)^m \left( \frac{B_t}{B_0} \right)^n \exp \left[ \frac{\left( m - m^2 \right) \sigma_d^2 t}{2} + \frac{\left( n - n^2 \right) \sigma_B^2 t}{2} - \rho mn \sigma_d \sigma_B t \right]
\]

where

\[
c' = \exp \left[ \frac{\left( m - m^2 \right) \sigma_d^2 t}{2} + \frac{\left( n - n^2 \right) \sigma_B^2 t}{2} - \rho mn \sigma_d \sigma_B t \right]
\]

This portfolio, in addition to the obvious delta exposure to $S_t$ and $B_t$, also has negative convexity to the movement of $S_t$ and $B_t$. Expanding the above to obtain the leading few terms and focusing on the case where $n + m = 1$ we obtain

\[
X_t = c' X_0 \left[ 1 + \frac{m}{S_0} \left( \frac{S_t}{S_0} - 1 \right) + n \left( \frac{B_t}{B_0} - 1 \right) + \frac{1}{2} m(m-1) \frac{\left( S_t/S_0 - B_t/B_0 \right)^2}{S_0} + \ldots \right]
\]

where

\[
c' = \exp \left[ \frac{\left( m - m^2 \right) \sigma_d^2 t}{2} + \frac{\left( n - n^2 \right) \sigma_B^2 t}{2} - \rho mn \sigma_d \sigma_B t \right]
\]

Comparing this with Eq 2, which is the corresponding expression for the single asset case, the two changes are that

- we now have a contribution from the bond return $\left( \frac{B_t}{B_0} - 1 \right)$, and
- the quadratic term has changed from $\left( \frac{S_t}{S_0} - 1 \right)^2$ to $\frac{\left( S_t/S_0 - B_t/B_0 \right)^2}{S_0}$.

Since we expect the bond return to be small in comparison to the equity return, this is entirely as expected and vindicates our earlier approximation.

For completeness, we show results from Monte Carlo simulations of the two-asset case. In this case we have modeled the stock and bond components as geometric Brownian motions with a volatilities of 20% and 6% respectively and -20% correlation.

These simulations confirm the earlier results showing increased worst returns and maximum drawdowns of the fixed weight rebalancing in comparison to the drift weights (Figure 18 and 19), and the ability of a momentum overlay to improve these risk characteristics (Figure 20 and 21).
Figure 18: Distribution of pay-off, one year, fixed weights (60/40) and drift weights (60/40) of two assets

Source: Man calculations, Monte Carlo simulation.

Figure 19: Distribution of worst drawdowns, one year, fixed weights (60/40) and drift weights (60/40) of two assets

Source: Man calculations, Monte Carlo simulation.

Figure 20: Terminal pay-off and worst drawdowns for fixed-weight rebalancing and momentum, 1-year horizon
A. Profit and loss
B. Maximum drawdown

Source: Man calculations, Monte Carlo simulation.

Figure 21: Terminal pay-off and worst drawdowns for fixed-weight rebalancing and momentum, 5-year horizon
A. Profit and loss
B. Maximum drawdown

Source: Man calculations, Monte Carlo simulation.
Important information
This material is communicated by AHL Partners LLP (‘AHL’ or the ‘Company’), a member of Man Group plc (‘Man’). AHL is authorised and regulated by the Financial Conduct Authority in the UK.

Hong Kong: To the extent this material is distributed in Hong Kong, this material is communicated by Man Investments (Hong Kong) Limited and has not been reviewed by the Securities and Futures Commission in Hong Kong. This material can only be communicated to intermediaries, and professional clients who are within one of the professional investors exemptions contained in the Securities and Futures Ordinance and must not be relied upon by any other person(s).

Singapore: To the extent this material is distributed in Singapore, it is for information purposes only and does not constitute any investment advice or research of any kind. This material can only be communicated to Institutional investors (as defined in Section 4A of the Securities and Futures Act, Chapter 289) and distributors/intermediaries and should not be relied upon by any other person(s).

Switzerland: To the extent this material is distributed in Switzerland, it is communicated by Man Investments AG, which is regulated by the Swiss Financial Market Authority FINMA.

Germany: To the extent this material is used in Germany, the distributing entity is Man (Europe) AG, which is authorised and regulated by the Liechtenstein Financial Market Authority (FINMA).

Australia: To the extent this material is distributed in Australia it is communicated by Man Investments Australia Limited, which is regulated by the Australian Securities & Investments Commission (ASIC). This information is for professional and sophisticated investors only.

United States: To the extent this material is distributed in the United States it is communicated by AHL Partners LLP, which is registered with the US Securities and Exchange Commission (‘SEC’) as an investment advisor and with the National Futures Association (‘NFA’), as authorized by the US Commodities Futures Trading Commission (‘CFTC’), as a commodity trading advisor and a commodity pool operator. The registrations and memberships described herein in no way imply that the SEC, CFTC or NFA have endorsed AHL to provide services.

This material is for informational purposes only and does not constitute any investment advice or research of any kind. This material can only be distributed to institutional, accredited investors and any other relevant person permitted to receive it.

Opinions expressed are those of the author and may not be shared by all personnel of Man. These opinions are subject to change without notice. This material is for information purposes only and does not constitute an offer or invitation to make an investment in any financial instrument or in any product to which any member of Man’s group of companies provides investment advisory or any other services. Any organisations, financial instruments or products described in this material are mentioned for reference purposes only and therefore, this material should not be construed as a commentary on the merits thereof or a recommendation for purchase or sale. Neither Man nor the author(s) shall be liable to any person for any action taken on the basis of the information provided. Any products and/or product categories mentioned may not be available in your jurisdiction or may significantly differ from what is available in your jurisdiction. Some statements contained in this material concerning goals, strategies, outlook or other non-historical matters may be forward-looking statements and are based on current indicators and expectations. These forward-looking statements speak only as of the date on which they are made, and Man undertakes no obligation to update or revise any forward-looking statements. These forward-looking statements are subject to risks and uncertainties that may cause actual results to differ materially from those contained in the statements. Past performance is not indicative of future results. This material is proprietary information of the Company and its affiliates and may not be reproduced or otherwise disseminated in whole or in part without prior written consent from the Company. The Company believes its data and text services to be reliable, but accuracy is not warranted or guaranteed. We do not assume any liability in the case of incorrectly reported or incomplete information. Information contained herein is provided from the Man database except where otherwise stated.

Past performance may not necessarily be repeated and is no guarantee or projection of future results. The value of investments and the income derived from those investments can go down as well as up. Future returns are not guaranteed and a loss of principal may occur.

Please note that all information in this document is furnished as of the date stated on the cover of this presentation, unless otherwise stated. This document and its contents are confidential and do not carry any right of publication or disclosure to any other party. All rights reserved, Man (2014).